

CONCLUSIONS AND PROSPECTS

It is appropriate at the conclusion of this work to make some attempt to summarize the main results that have been obtained, and to point out some areas that still need further clarification.

22.1 General conclusions

The work that has been reviewed in the previous chapters has involved the study of colliding plane waves. Such waves, however, must be considered only as convenient mathematical idealizations. Real waves are never plane. At a large distance from the source, it may be considered that an initially spherical wavefront may become approximately plane. This is reasonable. In this context, however, the assumption of plane symmetry involves not only the assumption that the curvature of the wavefront becomes negligible, but also the assumption that the wave is infinite in extent. It is this latter feature that must be considered to limit the applicability of the results obtained to real physical processes.

The field equations for colliding plane waves were initially presented above in Chapter 6 following the early work of Szekeres (1970, 1972) and Khan and Penrose (1971). The first exact solutions to be published were obtained using these equations. This whole approach to the subject, however, was altered by the work of Chandrasekhar and Ferrari (1984) and Chandrasekhar and Xanthopoulos (1985*a*), who showed that the field equations could also be written in a form similar to those for stationary axisymmetric space-times. It has been shown in Chapters 11 and 16 that the main field equations are identical to the well known Ernst equations. Once the equations are written in this form, a variety of generation techniques can be applied to obtain large classes of further solutions. Many explicit solutions have now been published, and have been reviewed above. It is also clear how further solutions may be obtained.

Initial conditions have been set up such that the approaching gravitational waves in regions II and III are described in terms of the components Ψ_4 and Ψ_0 respectively. These components continue into the interaction region, though with a modified amplitude and possibly phase. There is, however, an additional interactive gravitational component that always appears in the interaction region IV. This is described by the component Ψ_2 .

When electromagnetic waves collide, it appears that gravitational waves are always generated by the collision. These may be impulsive waves that occur along the boundaries of region IV only, or they may appear throughout the interaction region. However, they must always appear.

Any interpretation of the above results concerning the collision of plane waves must bear in mind the fact that the situation is highly idealized. Nevertheless, some general features concerning the interaction of waves in general relativity may be indicated. A similar situation is well known in cosmology. Here the general expansion and, particularly, the initial singularity of the Friedmann universes turn out to be general features of relativistic cosmologies, rather than particular consequences of the high degree of symmetry that is assumed.

In the case of colliding plane waves, the basic feature that has been substantially demonstrated above is the focusing effect of gravitational waves, and the slightly different focusing effects of waves of other types of matter. When two waves pass through each other, they will inevitably tend to focus each other. In the exact solutions that have been presented, the approaching waves are non-expanding. Thus, after the collision, the two waves will increasingly contract towards a focus. A wave crossing a gravitational wave will be focused astigmatically. Had the approaching waves been initially expanding, then it is reasonable to assume that, after the collision, their expansion would be slightly reduced. However, exact solutions describing such situations have not yet been obtained, although this general conclusion is supported by the work of Centrella and Matzner as described in Section 21.2.

When considering the collision of non-expanding plane waves, the waves after the collision will inevitably tend to a focus. This focus appears as a singularity in the space-time, and is one of the main features of the exact solutions that have been reviewed. This space-like singularity that appears in the interaction region is found to be either a scalar polynomial curvature singularity or, occasionally, a quasi-regular coordinate singularity forming a Killing–Cauchy horizon, according to the particular situation.

It has been argued above that, for arbitrary initial conditions, the singularity in the interaction region may normally be expected to be a curvature singularity, and therefore a boundary to the space-time. This appears to be the generic situation.

There is, however, a large class of exact solutions in which the curvature is bounded in the interaction region and the singularity appears to be merely a coordinate singularity. Such solutions include the degenerate Ferrari–Ibañez solution and the Chandrasekhar–Xanthopoulos solution

that are both of type D, the algebraically general Feinstein–Ibañez solution, and the equivalent electromagnetic solutions, including that of Bell and Szekeres. In these cases, the singularity can locally be removed by a coordinate transformation, and it appears to be possible to extend the space-time beyond this surface. However, any such extension must be non-unique. Further singularities also appear in these solutions, although they may be of a topological character. It has also been shown that these coordinate singularities are unstable with respect to at least one class of perturbations in the initial data. It follows that colliding plane waves will generically produce curvature singularities.

Boundaries to the space-time also occur in regions II and III. These prevent an observer from passing from the background region I, through region II or III, to region IV beyond what would then be a naked singularity. These boundaries are also quasi-regular coordinate singularities, and have loosely been described as ‘fold singularities’. These are closely related to the known global properties of plane waves.

These general features have largely been inferred from a consideration of one or two exact solutions. There is clearly scope for much further work, and we now turn to review the possibilities that are open.

22.2 Prospects for further work

Doubtless many more exact solutions describing colliding plane waves will be obtained in the next few years. However, there seems little point in simply generating more solutions of the same type just with a number of additional free parameters.

What would be much more significant would be to find a practical way to determine the solution in the interaction region for an arbitrary set of initial conditions. This would describe the interaction between two arbitrarily specified approaching waves. This problem has recently been addressed in a series of papers by Hauser and Ernst (1989*a,b*, 1990) as reviewed in Chapter 14. Further work is clearly required to develop practical methods to solve this problem.

It is also of great importance to further analyse the global structure of colliding plane wave solutions. At present the structure of only a few solutions are known in detail, and it has been demonstrated that the curvature singularity that occurs in most colliding plane wave solutions is in fact ‘generic’.

The structure of the Khan–Penrose solution has been thoroughly analysed by Matzner and Tipler (1984) as described in Section 8.2. The Bell–Szekeres solution has similarly been analysed by Clarke and Hayward (1989) and the degenerate Ferrari–Ibañez solution by Hayward (1989*a*).

The global structures of these particular cases are qualitatively different. A complete analysis has also been given of a solution of Babala (1987), but these are very much exceptions. A similarly thorough analysis of other classes of exact solutions would be most useful.

While commenting on global structure, it may also be mentioned that Hayward (1989*b*) has given an interesting interpretation of the time reverse of colliding plane wave solutions that contain Killing–Cauchy horizons rather than curvature singularities in terms of the snapping of cosmic strings.

Another topic that requires further attention is that of the solution-generation techniques that are appropriate for colliding plane waves. A large number of such techniques are already well known for stationary axisymmetric space-times. The relationships between these techniques have been investigated by Cosgrove (1980, 1982) and others. These techniques apply to all space-times that have two commuting Killing vectors. They therefore apply to stationary axisymmetric space-times, to cylindrically symmetric space-times and to colliding plane waves. In all these cases, the main field equations can be expressed as the same Ernst equation. However, the boundary conditions appropriate to each situation are significantly different. Generating techniques for stationary axisymmetric space-times have been adapted to space-times with two space-like commuting Killing vectors by Kitchingham (1984), but only in the context of cosmological solutions. It is particularly the significance of the difference in the boundary conditions and the physical significance of the different techniques that still requires further attention.

Indeed, it may be appropriate to consider further the relationship between exact solutions for stationary axisymmetric space-times, cylindrically symmetric space-times and colliding plane waves. Clearly some solutions can be interpreted in different ways, while others have only a single interpretation as they do not satisfy the boundary conditions for the alternative situations.

It has also been pointed out in Section 12.6 that it is possible to generate stationary axisymmetric solutions of Einstein’s equations by using twistor methods to construct self-dual Yang–Mills fields with appropriate symmetries (Ward 1983, Woodhouse 1987). The relation between this approach and other solution-generating techniques has been considered by Woodhouse and Mason (1988). However, it still remains to be demonstrated how twistor techniques may be developed to generate colliding plane wave solutions.

As again emphasized at the beginning of this chapter, in this monograph we have only been considering highly idealized situations in which exactly plane waves collide in a flat Minkowski background. These ideal-

izations constitute severe restrictions. To achieve any physical relevance, it will be necessary to show how the results obtained here can be extended to more realistic waves with curved wave fronts, and to collisions in non-flat backgrounds.

The collision of waves in an expanding background has been considered by Centrella and Matzner and described in Section 21.2. However, no exact solutions have been obtained. Although appropriate numerical techniques have now been developed and give interesting results, it would be very useful to have a number of analytic solutions. It is to be hoped that techniques may be developed that would enable such solutions to be obtained and thoroughly analysed.

Of even greater importance would be the presentation of exact solutions describing the collision of waves that did not have complete plane symmetry. A description of the collision of the more general class of *pp*-waves, for example, would be most interesting. It would then be possible to consider the collision of waves of finite energy, and of finite extent. However, it seems as though we are still a long way from obtaining exact solutions of this type, or even of setting up the appropriate initial conditions. In this context it may be noted that an initial study of the focusing properties of some *pp*-waves has been presented by Ferrari, Pendenza and Veneziano (1988).

An alternative approach has been suggested by Yurtsever (1988*b,d*), who has considered the collision of almost-plane waves and has argued that such collisions result in black hole type singularities surrounded by horizons. He has proved (Yurtsever, 1988*d*, 1989*a*) that colliding waves that are exactly plane symmetric across a region of finite (large) transverse size but which fall off in an arbitrary way at larger transverse distances inevitably produce singularities that have the same local structure as those for exact plane wave collisions. However, the global singularity structure for such situations and the existence of horizons are topics that require further investigation.

The related problem of the collision, or close encounter, of high-speed black holes has been considered in a substantial paper by D'Eath (1978). In the limit as the black holes approach the speed of light, the incoming gravitational fields in this case are concentrated in two plane-fronted shock regions. Aichelburg and Sexl (1971) have shown that the line element for the limiting case of a black hole moving with the speed of light is given by

$$ds^2 = 2dudr + 4\mu \log(X^2 + Y^2)du^2 - dX^2 - dY^2 \quad (22.1)$$

which is clearly a *pp*-wave of the form (4.1). It is thus of importance to consider the collision of two such waves, both for collisions and close

encounters in which the source is displaced from the origin of the X, Y coordinates.

It turns out to be possible, by considering how the far fields only are distorted and deflected by the collision, to estimate the amount of gravitational radiation that would be produced by such an encounter. The structure of the curved shocks after the collision has been analysed using perturbation methods. For certain values of the parameters, D'Eath has shown that a significant fraction of the collision energy can be radiated away as gravitational waves. A further analysis of this problem is most important.

Finally, it may be pointed out that all the calculations that have been described have involved classical theories. Exact solutions have been obtained of the classical theory of general relativity. However, in describing the collision of gravitational or electromagnetic waves, we are effectively considering macroscopic averages of graviton–graviton or photon–photon interactions. Ultimately, it will be necessary to consider quantum effects in such interactions. First steps in this direction have been taken by Lapedes (1977), who applied the Arnowitt–Deser–Misner quantization, and also by Yurtsever (1989*b*), who has considered linear quantum field theory on a Khan–Penrose colliding plane impulsive wave background. This seems to be a particularly important area for future research.

In conclusion, it should again be emphasized that the subject of this book, and the results reported in it, form only a single step in understanding the non-linearity that is inherent in Einstein's gravitational field equations.